Semester One Examination 2011
Question/Answer Booklet

MATHEMATICS SPECIALIST
MAS 3C/D
Section One
(Calculator Free)

Student name SOLUTIONS

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section One.
Formula sheet.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler.

Important note to candidates

No other items may be taken into the examination room. It is your responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor before reading any further.

Structure of this examination

	Number of questions	Working time (minutes)	Marks available
This Section (Section 1) Calculator Free	**8**	**50**	**40**
Section Two Calculator Assumed	13	100	80
		Total marks	120

Instructions to candidates

1. The rules for the conduct of WACE external examinations are detailed in the booklet *WACE Examinations Handbook*. Sitting this examination implies that you agree to abide by these rules.

2. Answer the questions in the spaces provided.

3. Show all working clearly. Any question, or part question, worth more than 2 marks requires valid working or justification to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

1. [4 marks]

Let $z = 5 - i$

(a) Find z^2 in the form $x + iy$. [1]

$z^2 = (5-i)^2 = 24 - 10i$ ✓

(b) Find $z + 2\bar{z}$ in the form $x + iy$. [1]

$z + 2\bar{z} = 5 - i + 2(5+i) = 15 + i$ ✓

(c) Find $\frac{i}{z}$ in the form $x + iy$. [2]

$\frac{i}{5-i} \times \frac{5+i}{5+i} = \frac{5i + i^2}{26}$ ✓

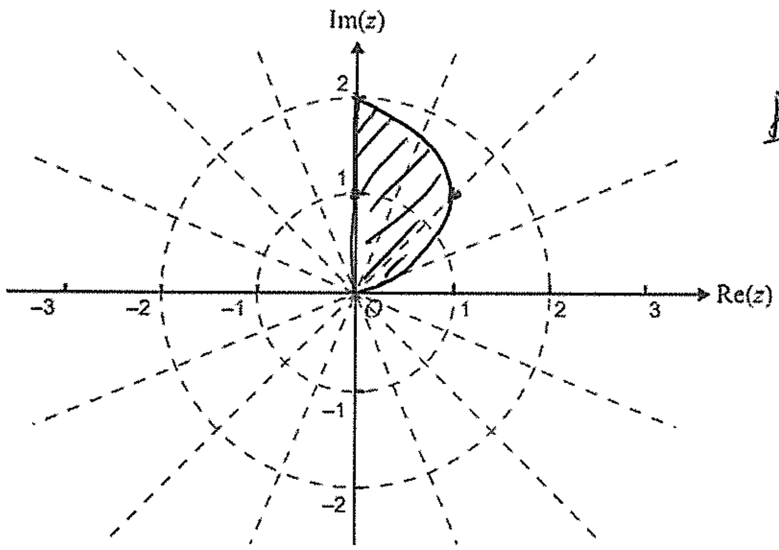
$= -\frac{1}{26} + \frac{5}{26}i$ ✓

$= \frac{1}{26}(-1 + 5i)$

2. [2 marks]

On the Argand diagram below, shade the region given by

$$\{z: |z - i| \leq 1\} \cap \{z: 0 \leq Arg\ z \leq \tfrac{\pi}{2}\}$$



▨ RR. ✓
half circle ✓

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3. [3 marks]

Solve the following equation for x ϵ **R**, giving an exact answer(s) for x.

$$4\,e^{2x-2} - 12\,e^{x-1} = 7$$

Let $e^{x-1} = y$

$\rightarrow \quad 4y^2 - 12y - 7 = 0$

$(2y-7)(2y+1) = 0$ ✓

$y = \frac{7}{2}, -\frac{1}{2}$

$\therefore \; e^{x-1} = \frac{7}{2}$ only ✓

$x - 1 = \ln\frac{7}{2}$

$x = 1 + \ln\frac{7}{2}$ ✓

4. [9 marks]

Find $\frac{dy}{dx}$ for the following

(a) $y = (e^x . x^{0.5})$ [1]

$\frac{dy}{dx} = e^x x^{\frac{1}{2}} + e^x . \frac{1}{2} x^{-\frac{1}{2}}$ ✓

$= e^x\left(x^{\frac{1}{2}} + \frac{1}{2x^{\frac{1}{2}}}\right)$

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(b) $y\sin\left(y^2\right)=2x$ [2]

$\frac{dy}{dx}\sin y^2 + y \cdot 2y\frac{dy}{dx}\cos y^2 = 2$ ✓

$\frac{dy}{dx}\left(\sin y^2 + 2y^2\cos y^2\right) = 2$

$\frac{dy}{dx} = \frac{2}{\sin y^2 + 2y^2\cos y^2}$ ✓

(c) $y=\ln\left(\frac{2\sin 3x}{2x^2+3}\right)$ $= \ln(2\sin 3x) - \ln(2x^2+3)$ ✓ [2]

$\frac{dy}{dx} = \frac{6\cos 3x}{2\sin 3x} - \frac{4x}{2x^2+3}$ ✓

$= \frac{3\cos 3x}{\sin 3x} - \frac{4x}{2x^2+3}$

(d) $x=\frac{1}{t+1}$, $y=2t+t^2$ Give your answer in terms of t. [2]

$\frac{dx}{dt} = \frac{-1}{(t+1)^2}$ ✓ $\frac{dy}{dt} = 2+2t = 2(1+t)$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -2(t+1)(t+1)^2$

$= -2(t+1)^3$ ✓

(e) For question (d) above find $\frac{d^2y}{dx^2}$ Give your answer in terms of t. [2]

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$ ✓

$= -6(t+1)^2 \cdot -(t+1)^2$

$= 6(t+1)^4$ ✓

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5. [6 marks]

If $z = 3 \text{ cis } (\frac{\pi}{4})$ and $w = 2 \text{ cis } (\frac{\pi}{3})$

Express the following in polar form.

(a) $\frac{z}{w} = \frac{3}{2} \text{ cis } \left(\frac{\pi}{4} - \frac{\pi}{3}\right)$ [1]

$= \frac{3}{2} \text{ cis } \left(-\frac{\pi}{12}\right)$ ✓

(b) $z\,w = 6 \text{ cis } \left(\frac{\pi}{4} + \frac{\pi}{3}\right)$ [1]

$= 6 \text{ cis } \left(\frac{7\pi}{12}\right)$ ✓

(c) $\bar{z} = 3 \text{ cis } \left(-\frac{\pi}{4}\right)$ ✓ [1]

(d) $iw = 2 \text{ cis } \left(\frac{\pi}{3} + \frac{\pi}{2}\right)$ [1]

$= 2 \text{ cis } \left(\frac{5\pi}{6}\right)$ ✓

(e) $w^3 = 2^3 \text{ cis } 3\left(\frac{\pi}{3}\right)$ [1]

$= 8 \text{ cis } \pi$ ✓

(f) $(\frac{w}{z})^6 = \dfrac{2^6 \text{ cis}\left(6 \cdot \frac{\pi}{3}\right)}{3^6 \text{ cis}\left(6 \cdot \frac{\pi}{4}\right)}$ [1]

$= \frac{64}{729} \text{ cis } \frac{\pi}{2}$ ✓

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6. [4 marks]

Determine the following integrals:

(a) $\int 3^{2x}\,dx = \int e^{2x\ln 3}\,dx$ [2]

$= \dfrac{e^{2x\ln 3}}{2\ln 3} + c$

$y = 3^{2x}$

$\ln y = 2x\ln 3$

$y = e^{2x\ln 3}$ ✓

(b) $\int \dfrac{4\sin x}{\cos^3 x}\,dx = \int 4\sin x\cos^{-3}x\,dx$ [2]

$= \dfrac{4\cos^{-2}x}{-2} + c$ ✓

$= -2\cos^{-2}x + c$ ✓

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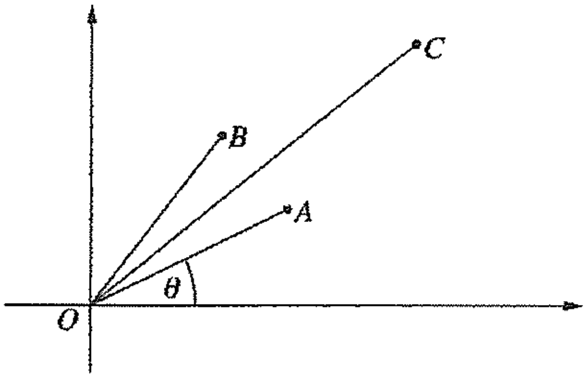
7. [5 marks]

Let $z = \cos\theta + i\sin\theta$ where $0 < \theta < \frac{\pi}{2}$.

On the Argand diagram the point A represents z, the point B represents z^2 and the point C represents $z + z^2$.



(a) Explain why the paralleogram OACB is a rhombus. [1]

parallelogram with adjacent sides congruent

$\Rightarrow \vec{OA} = \vec{BC} = \vec{OC} - \vec{OB}$ $\quad |\vec{OA}| = |z| = |\vec{OB}| = |z^2| = 1$

$\Rightarrow z = (z+z^2) - z^2 = z$

Simil. $\vec{OB} = \vec{AC}$ ✓

(b) Show that $\arg(z + z^2) = \frac{3\theta}{2}$. [1]

Since $\vec{OC}$ is the diagonal of a rhombus

$\vec{OC}$ bisects $\angle AOB \Rightarrow \angle COA = \theta/2$

hence $\arg(z+z^2) = \frac{\theta}{2} + \theta = \frac{3\theta}{2}$ ✓

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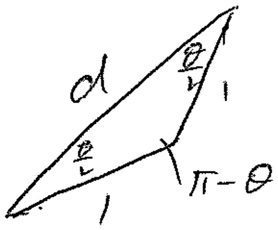
(c) Show that $|z + z^2| = 2\cos\frac{\theta}{2}$. Let $d = |z + z^2|$ [2]



Sine Rule

$$\frac{d}{\sin(\pi - \theta)} = \frac{1}{\sin\frac{\theta}{2}}$$

$$d = \frac{\sin\theta}{\sin\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= 2\cos\frac{\theta}{2}$$

(d) Using $z + z^2$, deduce that $\cos\theta + \cos 2\theta = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$ [1]

(b) and (c) $\Rightarrow$

$$z + z^2 = 2\cos\frac{\theta}{2}\left(\cos\frac{3\theta}{2} + i\sin\frac{3\theta}{2}\right)$$

$$LHS = \cos\theta + \cos 2\theta$$

$$RHS = 2\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad \text{(real part only)}$$

$$= LHS \quad \text{QED}$$

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8. [7 marks]

(a) Find the derivative of $y = x \ln x$ and hence or otherwise determine $\int \frac{3\ln x + 3}{x \ln x}\, dx$ [2]

$\frac{dy}{dx} = \ln x + 1$ ✓

$I = 3 \int \frac{\ln x + 1}{x \ln x}\, dx$

$= 3 \ln (x \ln x) + c$ ✓

(b) Evaluate exactly $\int_0^1 \frac{x}{\sqrt{1+x}}\, dx$ using the substitution $x = u^2 - 1$ [5]

$dx = 2u\, du$ ✓

$x = 1 \quad u = \sqrt{2}$ ✓

$x = 0 \quad u = 1$

$I = \int_1^{\sqrt{2}} \frac{u^2 - 1}{\sqrt{u^2}}\, 2u\, du$

$= 2 \int_1^{\sqrt{2}} (x^2 - 1)\, du$ ✓

$= 2 \left(\left[\frac{u^3}{3} - u \right]_1^{\sqrt{2}} \right)$ ✓

$= 2 \left[\left(\frac{2\sqrt{2}}{3} - \sqrt{2} \right) - \left(\frac{1}{3} - 1 \right) \right]$

$= \frac{-2\sqrt{2} + 4}{3}$ ✓

End of Questions

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